# Kinetic linear model of the interaction of helical magnetic perturbations with cylindrical plasmas

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The linear kinetic model of the interaction of helical rotating magnetic perturbations (resonant and nonresonant) with a tokamak plasma developed in [M. F. Heyn *et al.*, Nucl. Fusion **46**, S159 (2006)] is extended by a Galilean invariant collision operator and a different finite Larmor radius expansion scheme of particle current density. The model is applied to study the plasma screening effect on resonant magnetic perturbations and the resulting torques acting on the plasma at various orders of Larmor radius expansion. © *2011 American Institute of Physics*. [doi:10.1063/1.3551740]

## I. INTRODUCTION

The problem of the interaction of low frequency helical magnetic field perturbations with a plasma is an important issue for tokamaks where such perturbations are produced either on purpose, as in the case of ergodic divertors or magnetohydrodynamics (MHD) activity control coils, or without intention, as in the case of error fields due to small violations of the axisymmetry of the main magnetic coils.

A valuable method of edge localized mode (ELM) control using resonant magnetic perturbations (RMPs) has been proposed and successfully realized in DIII-D and JET tokamaks.[1,2] For an understanding and accurate modeling of the impact of RMPs on the plasma configuration, the amplitudes of these perturbations inside the plasma have to be known. While most theoretical studies of magnetic field ergodization and particle and heat transport use a vacuum perturbation field,[3–6] it has been shown in Refs. 7 and 8 that plasma response currents can cause a strong shielding of RMPs (by several orders of magnitude) and reduce the size of the ergodic layer at the edge.

The theoretical description of the interaction of RMPs with a plasma has a long history, see Refs. 9–18, and usually is based on various approximations within one- or two-fluid MHD. It has been verified in Ref. 8 that our kinetic model stays in approximate agreement about the RMP shielding with the reduced drift MHD model[17] except in regions with fast perpendicular electron rotation. This is a remarkable fact because for plasma parameters used there, MHD theory predicts that the resonant layer width is more than by an order of magnitude smaller than the ion Larmor radius and, therefore, MHD theory is formally invalid.

Our previous work[7,8] on the interaction of RMPs with a plasma is based on a linear kinetic model with a Krook collision operator and a finite Larmor radius expansion up to the first order. However, the Krook collision operator does not conserve the number of particles (charge) and this immediately leads to a violation of the Galilean covariance (with respect to a moving frame) of the results. It has been also observed that near resonance zones the radial scale of the RMP amplitudes is of the same order as the ion Larmor radius and, strictly speaking, the use of the Larmor radius expansion is at least questionable.

To verify the results of Refs. 7 and 8, the kinetic model is extended in the present work. The present model uses a charge conserving Fokker–Planck type collision operator and a more sophisticated finite Larmor radius expansion to higher orders. The results of the extended model when applied to study the screening of RMPs and torques acting on the plasma for a JET-like plasma configuration are presented and the differences to the former model are outlined.

The structure of the paper is as follows. In Sec. II, the kinetic equation is solved in noncanonical action-angle variables and the particle current density is obtained. In Sec. III, the shielding of RMPs is modeled. In Sec. IV, the results are summarized and discussed. Details of the extensive analytical calculations are given in three Appendices.

Throughout the paper, upper indices are used for contravariant and lower indices for covariant components of the vectors and summation over repeated skewed indices $A_\alpha B^\alpha$ is assumed. For this scalar product, the compact notation $\mathbf{A}\cdot\mathbf{B}$ is also sometimes used.

## II. MODEL DESCRIPTION

### A. Kinetic equation in action-angle variables

Let us consider the kinetic equation for the evolution of the particle distribution function $f(\mathbf{r},\mathbf{v},t)$ in phase space

$$\frac{\mathrm{d}}{\mathrm{d}t}f=\frac{\partial f}{\partial t}+\mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}}+\frac{\mathbf{F}}{m}\cdot\frac{\partial f}{\partial \mathbf{v}}=L_c f. \tag{1}$$

Here, $(\mathbf{r},\mathbf{v})$ are the position and the velocity of the particle, $\mathbf{F}=e[\mathbf{E}+(1/c)\mathbf{v}\times\mathbf{B}]$ is the Lorentz force acting on the particle, and $L_c$ is the collision operator.

Following the approach of Mahajan and Chen,[19] we rederive their kinetic equation in nonHamiltonian action-angle variables in a different and more compact way. In linear theory, the electromagnetic field can be represented as the sum of a constant background field and a small perturbation: $\mathbf{E}(\mathbf{r},t)=\mathbf{E}_0(\mathbf{r})+\widetilde{\mathbf{E}}(\mathbf{r},t)$, $\mathbf{B}(\mathbf{r},t)=\mathbf{B}_0(\mathbf{r})+\widetilde{\mathbf{B}}(\mathbf{r},t)$. Furthermore, it is assumed that the background electromagnetic field is axially symmetric and, therefore, there exist three independent invariants for the particle motion. The first task is to solve the kinetic equation of the unperturbed integrable system.

A convenient way to describe the particle motion in an integrable system is to use Hamiltonian formalism in action-angle variables. In such a set of canonical variables $\mathbf{J}=(J_1,J_2,J_3)$ and $\boldsymbol{\theta}=(\theta^1,\theta^2,\theta^3)$, the equations of particle motion are

$$\dot{\theta}^\alpha=\Omega^\alpha=\frac{\partial H_0(\mathbf{J})}{\partial J_\alpha},\quad \dot{J}_\alpha=-\frac{\partial H_0(\mathbf{J})}{\partial \theta^\alpha}=0. \tag{2}$$

Here, the Hamiltonian $H_0$ of the charged particle in the background electromagnetic field described by the vector potential $\mathbf{A}_0$ and scalar potential $\Phi_0$ is

$$H_0(\mathbf{p}_0,\mathbf{r})=(1/2)m_0[\mathbf{p}_0-(e/c)\mathbf{A}_0(\mathbf{r})]^2+e\Phi_0(\mathbf{r}), \tag{3}$$

with $e$ as the electric charge of the particle, $m_0$ as the mass of the particle, and $\mathbf{p}_0=m_0\mathbf{v}+(e/c)\mathbf{A}_0$ as the generalized momentum of the particle.

We suppose that the problem of the unperturbed particle motion can be solved completely: The invariants $\mathbf{J}$, the conjugate angles $\boldsymbol{\theta}$, and the generating function $G(\mathbf{r},\mathbf{J})$ that mediates the canonical transformation $(\mathbf{p}_0,\mathbf{r})\rightarrow(\mathbf{J},\boldsymbol{\theta})$ have been all obtained. In the new action-angle variables, the Hamiltonian Eq. (3) is a function of $\mathbf{J}$ solely and that is a major source of simplification in the subsequent analysis.

The kinetic equation for the background distribution function $f_0(\mathbf{p}_0,\mathbf{r})$ of the unperturbed system is

$$\mathbf{v}\cdot\frac{\partial f_0}{\partial \mathbf{r}}+\mathbf{F}_0\cdot\frac{\partial f_0}{\partial \mathbf{p}_0}=L_c f_0+S, \tag{4}$$

with $\mathbf{F}_0=e[\mathbf{E}_0+(1/c)\mathbf{v}\times\mathbf{B}_0]$ and $S$ representing internal sources and sinks.

A canonical transformation of variables from $(\mathbf{p}_0,\mathbf{r})$ to $(\mathbf{J},\boldsymbol{\theta})$ transforms Eq. (4) into

$$\mathcal{V}_0^\alpha\frac{\partial f_0}{\partial \theta^\alpha}+\mathcal{F}_{0\alpha}\frac{\partial f_0}{\partial J_\alpha}=L_c f_0+S, \tag{5}$$

where

$$\mathcal{V}_0^\alpha=\mathbf{v}\cdot\frac{\partial \mathbf{p}_0}{\partial J_\alpha}-\mathbf{F}_0\cdot\frac{\partial \mathbf{r}}{\partial J_\alpha},\quad \mathcal{F}_{0\alpha}=-\mathbf{v}\cdot\frac{\partial \mathbf{p}_0}{\partial \theta^\alpha}+\mathbf{F}_0\cdot\frac{\partial \mathbf{r}}{\partial \theta^\alpha}, \tag{6}$$

and it is used that the Jacobian of the canonical transformation $\partial(\mathbf{p}_0,\mathbf{r})/\partial(\mathbf{J},\boldsymbol{\theta})=1$.

In the Hamiltonian formulation, the kinetic equation for the background distribution function $f_0(\mathbf{J},\boldsymbol{\theta})$ has the following form,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}f_0&=\frac{\partial f_0}{\partial t}+\{f_0,H_0\}\\&=\frac{\partial f_0}{\partial \theta^\alpha}\frac{\partial H_0}{\partial J_\alpha}-\frac{\partial H_0}{\partial \theta^\alpha}\frac{\partial f_0}{\partial J_\alpha}\\&=\Omega^\alpha\frac{\partial f_0}{\partial \theta^\alpha}=L_c f_0+S.\end{aligned} \tag{7}$$

Equation (7) describes an exact dynamic equilibrium that is sustained by sources and sinks at the wall. For our purposes, it is sufficient to ignore small terms responsible for radial transport in the equilibrium. This means that we ignore the source term as well as small terms in $L_c\rightarrow L_c^{\mathrm{tr}}$ and assume that $f_0$ is Maxwellian, such that

$$\Omega^\alpha\frac{\partial f_0}{\partial \theta^\alpha}=L_c^{\mathrm{tr}}=0. \tag{8}$$

In this case, Eq. (8) implies that the background distribution function depends only on invariants of motion $f_0=f_0(\mathbf{J})$. In the present study, we will obtain results for a cylindrical model and in this case, Eq. (8) also allows for a drifting Maxwellian. From the comparison of Eqs. (7) and (5), it can be concluded that

$$\mathcal{V}_0^\alpha=\frac{\partial H_0}{\partial J_\alpha}=\Omega^\alpha,\quad \mathcal{F}_{0\alpha}=-\frac{\partial H_0}{\partial \theta^\alpha}=0. \tag{9}$$

Let us now consider the full kinetic Eq. (1) in the same $(\mathbf{J},\boldsymbol{\theta})$ variables as used for the description of the unperturbed motion,

$$\frac{\partial f}{\partial t}+\mathcal{V}^\alpha\frac{\partial f}{\partial \theta^\alpha}+\mathcal{F}_\alpha\frac{\partial f}{\partial J_\alpha}=L_c f, \tag{10}$$

where $\mathcal{V}^\alpha=\mathcal{V}_0^\alpha-\widetilde{\mathbf{F}}\cdot(\partial\mathbf{r}/\partial J_\alpha)$, $\mathcal{F}_\alpha=\mathcal{F}_{0\alpha}+\widetilde{\mathbf{F}}\cdot(\partial\mathbf{r}/\partial\theta^\alpha)$, and $\widetilde{\mathbf{F}}=e[\widetilde{\mathbf{E}}+(1/c)\mathbf{v}\times\widetilde{\mathbf{B}}]$ is the perturbation of the Lorentz force. In the perturbed system, these variables do not form a canonical conjugate pair with respect to the Hamiltonian, as pointed out by Mahajan and Chen.[19]

If the distribution function written as $f=f_0+\widetilde{f}$ is substituted into Eq. (10) and all terms proportional to the square of the perturbations are neglected, the linearized kinetic equation is obtained as

$$\frac{\partial \widetilde{f}}{\partial t}+\Omega^\alpha\frac{\partial \widetilde{f}}{\partial \theta^\alpha}+\dot{J}_\alpha\frac{\partial f_0}{\partial J_\alpha}=\hat{L}_c\widetilde{f},\quad \dot{J}_\alpha=\widetilde{\mathbf{F}}\cdot\frac{\partial \mathbf{r}}{\partial \theta^\alpha}, \tag{11}$$

where $\hat{L}_c$ is the linearized collision operator. This equation is just Eq. (20) of Ref. 19 with all nonlinear terms neglected and, instead, a collision term to the right-hand side of the equation has been added.

In the following, it is convenient to use vector and scalar potentials $(\widetilde{\mathbf{A}},\widetilde{\Phi})$ for the perturbation field so that $\widetilde{\mathbf{E}}=-\nabla\widetilde{\Phi}-(1/c)\partial\widetilde{\mathbf{A}}/\partial t$ and $\widetilde{\mathbf{B}}=\nabla\times\widetilde{\mathbf{A}}$. Making use of the relations

$$\nabla\widetilde{\Phi}\cdot\frac{\partial \mathbf{r}}{\partial \theta^\alpha}=\frac{\partial \widetilde{\Phi}}{\partial \theta^\alpha},$$

$$(\mathbf{v}\times\nabla\times\tilde{\mathbf{A}})\cdot\frac{\partial\mathbf{r}}{\partial\theta^{\alpha}}=\Omega^{\beta}\frac{\partial\tilde{\mathcal{A}}_{\beta}}{\partial\theta^{\alpha}}-\Omega^{\beta}\frac{\partial\tilde{\mathcal{A}}_{\alpha}}{\partial\theta^{\beta}}, \quad (12)$$

$$\tilde{\mathcal{A}}_{\alpha}=\tilde{\mathbf{A}}\cdot\frac{\partial\mathbf{r}}{\partial\theta^{\alpha}},$$

the time evolution for $J_\alpha$ is given by

$$\dot{J}_{\alpha}=-e\frac{\partial\tilde{\Phi}}{\partial\theta^{\alpha}}-\frac{e}{c}\left[\frac{\partial\tilde{\mathcal{A}}_{\alpha}}{\partial t}+\Omega^{\beta}\left(\frac{\partial\tilde{\mathcal{A}}_{\alpha}}{\partial\theta^{\beta}}-\frac{\partial\tilde{\mathcal{A}}_{\beta}}{\partial\theta^{\alpha}}\right)\right]. \quad (13)$$

This expression can also be directly obtained from Eqs. (11) and (A12) of Ref. 19.

Taking into account the cyclic nature of angle variables, one can expand all perturbed quantities as Fourier series in $\boldsymbol{\theta}$, e.g.,

$$\tilde{f}(\mathbf{J},\boldsymbol{\theta},t)=\sum_{\mathbf{m}}\tilde{f}_{\mathbf{m}}(\mathbf{J},t)e^{i\mathbf{m}\cdot\boldsymbol{\theta}}. \quad (14)$$

Substituting Eq. (14) in Eq. (11), we obtain the linear kinetic equation for the Fourier amplitudes $\tilde{f}_{\mathbf{m}}(\mathbf{J},t)$

$$\frac{\partial\tilde{f}_{\mathbf{m}}}{\partial t}+i\mathbf{m}\cdot\boldsymbol{\Omega}\tilde{f}_{\mathbf{m}}-\hat{L}_c\tilde{f}_{\mathbf{m}}=\tilde{Q}_{\mathbf{m}}, \quad (15)$$

where the source term $\tilde{Q}_{\mathbf{m}}$ has been defined as

$$\tilde{Q}_{\mathbf{m}}(\mathbf{J},t)=e\frac{\partial f_0}{\partial J_{\alpha}}\left\{im_{\alpha}\tilde{\Phi}_{\mathbf{m}}+\frac{1}{c}\left\{\frac{\partial(\tilde{\mathcal{A}}_{\alpha})_{\mathbf{m}}}{\partial t}+i\Omega^{\beta}[m_{\beta}(\tilde{\mathcal{A}}_{\alpha})_{\mathbf{m}}-m_{\alpha}(\tilde{\mathcal{A}}_{\beta})_{\mathbf{m}}]\right\}\right\}, \quad (16)$$

with $\tilde{\Phi}_{\mathbf{m}}$ and $(\tilde{\mathcal{A}}_{\alpha})_{\mathbf{m}}$ as the Fourier amplitudes of the electric and vector potential of the perturbation.

Let us introduce a set of curvilinear space coordinates $x=(x^1,x^2,x^3)$ with metric tensor $g_{ij}$ and the metric determinant denoted as $g$. The contravariant components of the perturbation of particle current density can conveniently be written as a phase space integral with a delta-function

$$\tilde{j}^{k}(\mathbf{x},t)=e\int d^3p_0 v^k\tilde{f}=\frac{e}{\sqrt{g}}\int d^3\theta\int d^3J\,\delta[\mathbf{x}-\mathbf{x}_c(\mathbf{J},\boldsymbol{\theta})]\times v^k(\mathbf{J},\boldsymbol{\theta})\tilde{f}(\mathbf{J},\boldsymbol{\theta},t), \quad (17)$$

where $\mathbf{x}_c(\mathbf{J},\boldsymbol{\theta})$ are curvilinear coordinates expressed as functions of action-angle variables and $v^k=(\partial x_c^k/\partial\theta^{\alpha})\Omega^{\alpha}$ is the particle velocity. In contrast to Eq. (11) of Ref. 7, the integration is performed over the unperturbed generalized momentum.

## B. Unperturbed particle motion

In the following, the interaction of helical magnetic perturbations with a tokamak plasma is considered in the geometry of a straight periodic cylinder of length $L=2\pi R$ with rotational transform of the background magnetic field using cylindrical coordinates $\mathbf{x}=(r,\vartheta,z)$. We introduce the parallel $C_{\parallel}=\mathbf{C}\cdot\mathbf{h}$ and the perpendicular $C_{\perp}=\mathbf{C}\cdot\mathbf{e}_{\perp}$ projections of a vector, where $\mathbf{h}=\mathbf{B}_0/B_0$, $\mathbf{B}_0=\nabla\times\mathbf{A}_0$, and $\mathbf{e}_{\perp}=\mathbf{h}\times\mathbf{e}_r$.

In cylindrical geometry, it is simple to perform a canonical transformation to action-angle variables for the unperturbed motion and the particle trajectories $\mathbf{x}(t)=\mathbf{x}_c(\boldsymbol{\theta}(t),\mathbf{J})$ can be represented as[7]

$$x_c^i(\boldsymbol{\theta},\mathbf{J})=x_g^i(\boldsymbol{\theta},\mathbf{J})+\rho^i(\boldsymbol{\theta},\mathbf{J}), \qquad \int_{-\pi}^{\pi}d\phi\,\rho^i(\boldsymbol{\theta},\mathbf{J})=0, \quad (18)$$

where the guiding center coordinates, $\mathbf{x}_g(\boldsymbol{\theta},\mathbf{J})=[r_g(\mathbf{J}),\vartheta_g,z_g]$, are independent of the gyrophase $\phi$. The canonical angles are $\boldsymbol{\theta}=(\phi,\vartheta_g,z_g)$ being the gyrophase, azimuth and $z$-coordinate of the guiding center, respectively. The canonical actions $\mathbf{J}=(J_{\perp},P_{\vartheta},P_z)$ are the perpendicular adiabatic invariant $J_{\perp}\approx m_0v_{\perp}^2/(2\omega_c)$ and the covariant $(\vartheta,z)$ components of the generalized momentum.

In lowest order over thermal motion, the unperturbed orbits [Eq. (18)] are described by

$$\rho^r=-\rho\cos\phi, \quad \rho^{\vartheta}=\frac{\rho}{r_0}h_z\sin\phi, \quad \rho^z=-\frac{\rho}{r_0}h_{\vartheta}\sin\phi, \quad (19)$$

$$\Omega^{\phi}=\omega_c, \quad \Omega^{\vartheta}=h^{\vartheta}u_{\parallel}+V_E^{\vartheta}, \quad \Omega^z=h^zu_{\parallel}+V_E^z, \quad (20)$$

where $\omega_c=eB_0/(m_0c)$, $\rho=[2J_{\perp}/(m_0\omega_c)]^{1/2}$, $\mathbf{V}_E=(c\Phi_0'/B_0)\mathbf{e}_{\perp}$, and all functions of coordinates are taken at $r=r_0$. Here, for convenience, the new variables $r_0$ (radial guiding center position) and $u_{\parallel}$ (parallel particle velocity) have been introduced through

$$\mathbf{P}=\mathbf{P}(r_0,u_{\parallel})=m_0[\mathbf{h}(r_0)u_{\parallel}+\mathbf{V}_E(r_0)]+\frac{e}{c}\mathbf{A}_0(r_0). \quad (21)$$

Although it is straightforward to obtain higher order Larmor radius corrections to the unperturbed orbits [Eqs. (19) and (20)], we neglect all particle drifts related to the inhomogeneity of the background magnetic field and keep only the electric drift of particles in the present model.

## C. Solution of the kinetic equation

Particle collisions in the kinetic Eq. (15) are modeled by a one-dimensional Fokker–Planck collision operator (Ornstein–Uhlenbeck approximation, see Ref. 20)

$$\hat{L}_c\tilde{f}(\mathbf{J})=\frac{\partial}{\partial u_{\parallel}}D\left[\frac{\partial}{\partial u_{\parallel}}+\frac{u_{\parallel}-V_{\parallel}}{v_T^2}\right]\tilde{f}(\mathbf{J}), \quad (22)$$

where $D$ is a constant diffusion coefficient in velocity space, $v_T=\sqrt{T_0/m_0}$ is the thermal velocity, and $V_{\parallel}$ is a bulk parallel velocity of the given species.

The collision operator [Eq. (22)] conserves the number of particles but it does not conserve the momentum and energy of species. Another property of the collision operator, $\hat{L}_cf_0=0$, is also satisfied for the operator [Eq. (22)] and the background distribution function Eq. (B1). The collision operator [Eq. (22)] essentially improves our previous model[7,8] because it ensures the invariance of the distribution function $\tilde{f}(\mathbf{J},\boldsymbol{\theta},t)$ with respect to Galilean transformations to a moving frame.

Let us consider the time evolution [Eq. (15)] of the Fourier amplitudes of the distribution function with $\mathbf{m}=(l,k_\vartheta,k_z)$. Introducing the new velocity variable, $u=u_\parallel-V_\parallel$, the kinetic Eq. (15) takes the form

$$\left[\frac{\partial}{\partial t}+i(\omega_l+k_\parallel u)-D\frac{\partial}{\partial u}\left(\frac{\partial}{\partial u}+\frac{u}{v_t^2}\right)\right]\tilde{f}_{\mathbf{m}}(u,t)=\tilde{Q}_{\mathbf{m}}(u,t), \tag{23}$$

where $k_\parallel=k_\vartheta h^\vartheta+k_z h^z$ and $k_\perp=(h_z k_\vartheta-h_\vartheta k_z)/r_0$ are parallel and perpendicular wave numbers and $\omega_E=k_\perp V_E$ is a frequency of electric particle drift, $\omega_l=l\omega_c+\omega_E+k_\parallel V_\parallel$.

The tedious details of obtaining the solution can be found in Appendix A. The solution of the kinetic equation can be written as

$$\tilde{f}_{\mathbf{m}}(u,t)=\int_0^{t-t_0}\mathrm{d}\tau\int_{-\infty}^{+\infty}\mathrm{d}u'\,\hat{G}(u,u',\tau)\tilde{Q}_{\mathbf{m}}(u',t-\tau), \tag{24}$$

with Green's function

$$\hat{G}(u,u',\tau)=\frac{1}{\sqrt{4\pi\tilde{a}}}\exp\left\{i\frac{k_\parallel}{\nu}[u-u'] -c-\frac{1}{4\tilde{a}}[u-u'\mathrm{e}^{-\nu\tau}+i\tilde{b}]^2\right\}, \tag{25}$$

and

$$\tilde{a}(\tau)=\frac{v_T^2}{2}[1-\mathrm{e}^{-2\nu\tau}],\quad \tilde{b}(\tau)=\frac{2k_\parallel v_T^2}{\nu}[1-\mathrm{e}^{-\nu\tau}],$$
$$c(\tau)=\left[i\omega_l+\frac{k_\parallel^2 v_T^2}{\nu}\right]\tau, \tag{26}$$

where $\nu=D/v_T^2$ is the collision frequency. In the collisionless limit $\nu\to 0$, the solution [Eqs. (24)–(26)] reduces to the well-known result for $\tilde{f}_{\mathbf{m}}$ (see Eq. (28) of Ref. 19).

The Green's function has an important property,

$$\hat{G}(u,u',\tau)\mathrm{e}^{-(1/2)(u'/v_T)^2}=\hat{G}(u',u,\tau)\mathrm{e}^{-(1/2)(u/v_T)^2}, \tag{27}$$

a feature which is extensively used in the following.

### D. Evaluation of the current density

Let us evaluate the perturbation of current density produced in a plasma by a single harmonic perturbation of the vector potential with the frequency $\omega$

$$\tilde{\mathbf{A}}(\mathbf{x},t)=\mathrm{Re}\,\tilde{\mathbf{A}}(r)\mathrm{e}^{ik_\vartheta\vartheta+ik_z z-i\omega t}, \tag{28}$$

where $k_\vartheta=m$, $k_z=n/R$, and $(m,n)$ are poloidal and toroidal numbers of the helical perturbation. For the perturbation field, we use the radiation gauge $\tilde{\Phi}=0$ in which the electric field is defined solely by the vector potential $\tilde{\mathbf{E}}=\frac{i\omega}{c}\tilde{\mathbf{A}}$. Nevertheless, we can also apply our approach to static perturbations with $\omega=0$ by a transition to a moving frame where the perturbation frequency is finite.

For the perturbation field [Eq. (28)], the source term [Eq. (16)] is

$$\tilde{Q}_{\mathbf{m}}(\mathbf{J})=-\frac{e}{\omega}(\tilde{\mathcal{E}}_\alpha)_{\mathbf{m}}\left[\left(\mathbf{m}\cdot\frac{\partial f_0}{\partial\mathbf{J}}\right)\Omega^\alpha-(\mathbf{m}\cdot\mathbf{\Omega}-\omega)\frac{\partial f_0}{\partial J_\alpha}\right]. \tag{29}$$

The Fourier amplitudes of the "canonical" electric field,

$$\tilde{\mathcal{E}}_\alpha(\mathbf{J},\boldsymbol{\theta},t)\equiv\frac{\partial x_c^j}{\partial\theta^\alpha}\tilde{E}_j=\left[\frac{\partial x_g^j}{\partial\theta^\alpha}+\frac{\partial\rho^j}{\partial\theta^\alpha}\right]\tilde{E}_j(r_g+\rho^r)\times\mathrm{e}^{i(k_\vartheta\vartheta_g+k_z z_g+\mathbf{k}\cdot\boldsymbol{\rho}-\omega t)}, \tag{30}$$

with $\mathbf{k}=(0,k_\vartheta,k_z)$, $\boldsymbol{\rho}=(\rho^r,\rho^\vartheta,\rho^z)$, and drift representation [Eq. (18)] of the orbits, can be evaluated by a Taylor series expansion of the electric field amplitude $\tilde{E}_j(r_g+\rho^r)$ if it does not change substantially over the scale of the particle gyration.

For $\tilde{\mathcal{E}}_\alpha$, we introduce the finite Larmor radius expansion of order $N$ as follows:

$$\tilde{\mathcal{E}}_\alpha^{(N)}(\mathbf{J},\boldsymbol{\theta},t)=\mathrm{e}^{i(k_\vartheta\vartheta_g+k_z z_g-\omega t)}\left(\sum_{n=0}^N a_\alpha^j(n)\frac{\partial^n}{\partial r_g^n}\right)\tilde{E}_j(r_g), \tag{31}$$

where

$$a_\alpha^j(n)=\left[\frac{\partial x_g^j}{\partial\theta^\alpha}+\frac{\partial\rho^j}{\partial\theta^\alpha}-\delta_{nN}\delta_r^j\frac{\partial\rho^r}{\partial\theta^\alpha}\right]\frac{(\rho^r)^n}{n!}\mathrm{e}^{i\mathbf{k}\cdot\boldsymbol{\rho}}. \tag{32}$$

In contrast to the expansion used in our previous model,[7,8] only the radial dependence of the electric field is expanded while the angle dependence is treated without approximation. The important property of the expansion scheme, namely the preservation of gradients for the expanded quantities, i.e., if $\tilde{\mathbf{E}}=-\nabla\tilde{\Phi}$ then $\tilde{\mathcal{E}}_\alpha^{(N)}=-\partial\tilde{\Phi}^{[N]}/\partial\theta^\alpha$, is guaranteed in the expansion Eqs. (31) and (32) by the term with Kronecker deltas. This property leads to the correct zero frequency limit of $\tilde{f}_{\mathbf{m}}^{(N)}$ and also ensures its covariance with respect to Galilean transformations to a moving frame.

Making use of Eqs. (14), (31), and (32), the Fourier amplitude of $\tilde{\mathcal{E}}_\alpha^{(N)}$ with $\mathbf{m}'=(l,m_\vartheta,m_z)$ is

$$(\tilde{\mathcal{E}}_\alpha^{(N)})_{\mathbf{m}'}=\delta_{m_\vartheta,k_\vartheta}\delta_{m_z,k_z}\mathrm{e}^{-i\omega t}\left(\sum_{n=0}^N[a_\alpha^j(n)]_l\frac{\partial^n}{\partial r_g^n}\right)\tilde{E}_j(r_g), \tag{33}$$

where $[a_\alpha^j(n)]_l$ is the Fourier transform of the matrix $a_\alpha^j(n)$ over gyrophase and $l$ is the index of the cyclotron harmonic.

Taking into account Eqs. (14), (24), (29), and (33), the perturbation of the distribution function is

$$\tilde{f}^{(N)}(\mathbf{J},\boldsymbol{\theta},t)=\frac{ie}{\omega}\mathrm{e}^{ik_\vartheta\vartheta_g+ik_z z_g-i\omega t}\sum_l\mathrm{e}^{il\phi}f_l^\beta\left(\sum_{n=0}^N[a_\beta^j(n)]_l\frac{\partial^n}{\partial r_g^n}\right)\times\tilde{E}_j(r_g), \tag{34}$$

$$f_l^\beta=i\int_0^{+\infty}\mathrm{d}\tau\int_{-\infty}^{+\infty}\mathrm{d}u'\,\hat{G}(u,u',\tau)\times\mathrm{e}^{i\omega\tau}\left[\left(\mathbf{m}\cdot\frac{\partial f_0}{\partial\mathbf{J}}\right)\Omega^\beta-(\mathbf{m}\cdot\mathbf{\Omega}-\omega)\frac{\partial f_0}{\partial J_\beta}\right]_{u'}, \tag{35}$$

where we have set $t_0=-\infty$ for harmonic perturbations, $\mathbf{m}=(l,k_\vartheta,k_z)$, and the factor $\mathrm{e}^{i\omega\tau}$ is naturally included in

Green's function [Eq. (25)], replacing $c(\tau)\rightarrow\tilde{c}(\tau)=c(\tau)-i\omega\tau$ (below, we denote the redefined Green's function as $\tilde{G}$). The notation $[\dots]_{u'}$ means the evaluation of the shifted parallel velocity at $u'$.

The next step is to substitute the expression [Eq. (34)] in the phase space integral

$$\int d^3\theta\int d^3J=\int d\phi d\vartheta_g dz_g\int dJ_\perp dr_0 du_\parallel\frac{\partial(P_\vartheta,P_z)}{\partial(r_0,u_\parallel)} \quad (36)$$

for the current density [Eq. (17)]. Integrations over $\vartheta_g$ and $z_g$ are trivial, and integration over gyrophase $\phi$ can be performed by the same (formal) Larmor radius expansion of the delta-function as used for the electric field

$$\left[\frac{\partial x_c^k}{\partial\theta^\alpha}\delta(r-r_g-\rho^r)\right]^{(N)}=\left[\frac{\partial x_g^k}{\partial\theta^\alpha}+\frac{\partial\rho^k}{\partial\theta^\alpha}-\delta_{nN}\delta_r^k\frac{\partial\rho^r}{\partial\theta^\alpha}\right]\times\sum_{n'=0}^{N}(-)^{n'}\frac{(\rho^r)^{n'}}{n'!}\frac{\partial^{n'}}{\partial r^{n'}}\delta(r-r_g), \quad (37)$$

so that

$$\int_{-\pi}^{\pi}d\phi e^{il\phi-i\mathbf{k}\cdot\boldsymbol{\rho}}\frac{\partial x_c^k}{\partial\theta^\alpha}\delta(r-r_g-\rho^r)\approx 2\pi\sum_{n'=0}^{N}[a_\alpha^k(n')]_l^*(-)^{n'}\frac{\partial^{n'}}{\partial r^{n'}}\delta(r-r_g). \quad (38)$$

After the trivial integration over $r_0$ (that is, the gyrocenter radius $r_g$ in our approximation) with the delta-function, the perturbation of the current density becomes

$$\tilde{j}^k_{(N)}=\frac{1}{r}\sum_{n,n'=0}^{N}(-)^n\frac{\partial^n}{\partial r^n}\left(r\sigma_{nn'}^{kj}(r,\mathbf{k})\frac{\partial^{n'}}{\partial r^{n'}}\tilde{E}_j\right), \quad (39)$$

where conductivity matrices have been defined as

$$\sigma_{nn'}^{kj}(r,\mathbf{k})=\frac{2\pi ie^2}{r\omega}\sum_{l=-\infty}^{+\infty}\int dJ_\perp du_\parallel\frac{\partial(P_\vartheta,P_z)}{\partial(r_0,u_\parallel)}[a_\alpha^k(n)]_l^*\times\Omega^\alpha[a_\beta^j(n')]_l f_l^\beta|_{r_0=r}. \quad (40)$$

The perturbation of the plasma current density depends on the electric field and its radial derivatives up to the $(2N)$-th order. In cylindrical geometry, only the Larmor gyration effect can couple the electric field to the current density nonlocally in the radial variable.

### E. Evaluation of the conductivity matrices $\sigma_{nn'}^{kj}$

The Fourier transform of the matrix $a_\beta^j(n')$ over gyrophase is obtained from Eq. (32), the explicit particle orbits [Eq. (19)], and the relations $\mathbf{k}\cdot\boldsymbol{\rho}=k_\perp\rho\sin\phi$ and $\exp(i\kappa\rho\sin\phi)=\Sigma_l J_l(\kappa\rho)e^{il\phi}$

$$[a(n')]_l=\frac{1}{n'!}\begin{pmatrix}-i(1-\delta_{n'N})\frac{\partial}{\partial\kappa_1} & 0 & 0\\ -i\frac{h_z}{r_0}\frac{\partial}{\partial\kappa_2} & 1 & 0\\ i\frac{h_\vartheta}{r_0}\frac{\partial}{\partial\kappa_2} & 0 & 1\end{pmatrix}\times i^{n'}\frac{\partial^{n'}}{\partial\kappa_2^{n'}}J_l(\kappa\rho)e^{il\xi}|_{(\kappa_1=k_\perp,\kappa_2=0)}, \quad (41)$$

with $\kappa=\sqrt{\kappa_1^2+\kappa_2^2}$, $\xi=\arctan(\kappa_2/\kappa_1)$, and the Bessel function $J_l(\kappa\rho)$. For the following, it is convenient to cast matrix $[a(n')]_l$ in compact form

$$[a_\beta^j(n')]_l=\frac{i^{n'}}{n'!}A_\beta^j(n')i^{Q_\beta^j+R_\beta^j}\frac{\partial^{Q_\beta^j}}{\partial\kappa_1^{Q_\beta^j}}\frac{\partial^{R_\beta^j}}{\partial\kappa_2^{R_\beta^j}}\frac{\partial^{n'}}{\partial\kappa_2^{n'}}J_l(\kappa\rho)e^{il\xi}|_{(\kappa_1=k_\perp,k_2=0)}, \quad (42)$$

with

$$A(n')=\begin{pmatrix}\delta_{n'N}-1 & 0 & 0\\ -\frac{h_z}{r_0} & 1 & 0\\ \frac{h_\vartheta}{r_0} & 0 & 1\end{pmatrix},\quad Q=\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix},\quad R=\begin{pmatrix}0&0&0\\1&0&0\\1&0&0\end{pmatrix}, \quad (43)$$

and $j$ ($\beta$) numerates rows (columns) of the matrices.

In Appendix B, the following presentations are derived,

$$\frac{\partial(P_\vartheta,P_z)}{\partial(r_0,u_\parallel)}\Omega^\alpha=\sum_\eta(u)^\eta I_\eta^\alpha, \quad (44)$$

$$\left[\left(\mathbf{m}\cdot\frac{\partial f_0}{\partial\mathbf{J}}\right)\Omega^\beta-(\mathbf{m}\cdot\boldsymbol{\Omega}-\omega)\frac{\partial f_0}{\partial J_\beta}\right]_{u'}=-\left(\frac{f_0}{T_0}\right)_{u'}\sum_{\mu,\nu}(u')^\mu(J_\perp)^\nu F_{\mu\nu}^\beta, \quad (45)$$

where coefficients $I_\eta^\alpha, F_{\mu\nu}^\beta$ are defined in Eqs. (B3) and (B16)–(B19) and the unperturbed distribution function $f_0$ is used in the form of an inhomogeneous drifting Maxwellian Eq. (B1).

Substituting Eqs. (44) and (45) in Eq. (40) and separating integrations over $J_\perp$ and $u$, we finally get

$$\sigma_{nn'}^{kj}=\frac{2\pi ie^2}{r\omega}\frac{(-)^{n+1}}{(2\pi)^{3/2}}\frac{i^{n+n'+1}}{n!n'!}\frac{n_0}{m_0^4v_T^5}\times\sum_{\alpha,\beta}(-)^{S_\alpha^k}i^{S_\alpha^k+S_\beta^j}A_\alpha^k(n)A_\beta^j(n')\times\sum_\eta I_\eta^\alpha\sum_{\mu,\nu}\sum_l F_{\mu\nu}^\beta W_l^{\eta\mu}D_{\nu l}^{Q_\alpha^k;R_\alpha^k+n;Q_\beta^j;R_\beta^j+n'}, \quad (46)$$

where matrix $S \equiv Q+R$, and we have defined two special functions,

$$W_l^{mn} \equiv \int_0^{+\infty} \mathrm{d}\tau \int_{-\infty}^{+\infty} \mathrm{d}u \int_{-\infty}^{+\infty} \mathrm{d}u' \tilde{G}(u,u',\tau) \times \mathrm{e}^{-(1/2)(u'/v_T)^2}(u)^m(u')^n, \quad (47)$$

and

$$D_{\nu l}^{m;n;m';n'} \equiv 2^{-\nu}(m_0\omega_c)^{\nu+1} \frac{\partial^m}{\partial\tilde{\kappa}_1^m} \frac{\partial^n}{\partial\tilde{\kappa}_2^n} \frac{\partial^{m'}}{\partial\kappa_1^{m'}} \frac{\partial^{n'}}{\partial\kappa_2^{n'}} \mathrm{e}^{il(\xi-\tilde{\xi})} \times \int_0^{+\infty} \mathrm{d}\rho \mathrm{e}^{-(1/2)(\rho/\rho_L)^2}\rho^{2\nu+1} \times J_l(\tilde{\kappa}\rho)J_l(\kappa\rho)\big|_{\{\tilde{\kappa}_1=k_\perp,\tilde{\kappa}_2=0,\kappa_1=k_\perp,\kappa_2=0\}}, \quad (48)$$

where $\tilde{\kappa}=\sqrt{\tilde{\kappa}_1^2+\tilde{\kappa}_2^2}$, $\tilde{\xi}=\arctan(\tilde{\kappa}_2/\tilde{\kappa}_1)$, $\rho^2=2J_\perp/(m_0\omega_c)$, and $\rho_L=v_T/\omega_c$.

From the Green's function property [Eq. (27)], it follows that the $W$-function is symmetric with respect to a permutation of the upper indices. Its values depend on the cyclotron harmonic index $l$ because the Green's function [Eqs. (25) and (26)] depends on $\omega_l=l\omega_c+\omega_E+k_\parallel V_\parallel$. The $D$-function becomes complex conjugate under a permutation of $m$ and $m'$ or $n$ and $n'$. Both introduced special functions are evaluated analytically, as shown in Appendix C. The $W$-function can be reduced to a combination of confluent hypergeometric functions of the first kind [Eq. (C1)] while the $D$-function can be expressed in terms of modified Bessel functions of the first kind [Eq. (C5)].

### F. Wave equations

The perturbed electromagnetic field is a solution of Maxwell's equations,

$$\nabla\times\tilde{\mathbf{E}} = \frac{i\omega}{c}\tilde{\mathbf{B}}, \quad \nabla\times\tilde{\mathbf{B}} = -\frac{i\omega}{c}\tilde{\mathbf{E}} + \frac{4\pi}{c}(\tilde{\mathbf{j}}_p+\tilde{\mathbf{j}}_a), \quad (49)$$

where $\tilde{\mathbf{j}}_a$ is an antenna current density and $\tilde{\mathbf{j}}_p$ is a plasma (electrons+ions) current density which can be evaluated by Eqs. (39) and (46). In cylindrical geometry and for a single harmonic perturbation [Eq. (28)], these equations reduce to a set of ordinary differential over the radial variable for the field amplitudes. For the $N$th order Larmor radius expansion in the particle current density, the number of independent free solutions ("waves") of Maxwell's equations is $6N-2$.

A typical set of boundary conditions assumes that the plasma is surrounded by an ideal metallic wall located at $r=r_w$. The antenna is modeled by an infinitely thin divergence free current density flowing at $r=r_a<r_w$. In a more elaborated version of the settings, instabilities can also be studied, e.g., resistive wall modes.

The stiff set of equations is solved numerically using a reorthonormalization procedure.

### G. Power absorption and torques

Let us check that the total power absorption is nonnegative for the case of a Boltzmann distribution of the background particles, $f_0(\mathbf{J})=f_0(H_0)$. In that case,

$$\frac{\partial f_0}{\partial J_\beta} = \Omega^\beta\frac{\partial f_0}{\partial H_0} = -\frac{\Omega^\beta}{T_0}f_0. \quad (50)$$

First, the current density [Eq. (17)] is substituted into the expression for the power absorbed in the whole plasma volume by a specific sort of particles,

$$P_{\mathrm{tot}}^{(N)} = \frac{1}{2}\mathrm{Re}\int \mathrm{d}^3\mathrm{x}\sqrt{g}\tilde{j}_{(N)}^k\tilde{E}_k^*, \quad (51)$$

then Eqs. (14), (24), and (29) are used as well as the relation

$$\int \mathrm{d}^3\mathrm{x}\tilde{E}_k^*(\mathbf{x})\left[\frac{\partial \mathrm{x}_c^k}{\partial\theta^\alpha}\delta(\mathbf{x}-\mathbf{x}_c)\right]^{(N)} = (\tilde{\mathcal{E}}_\alpha^{(N)})^*, \quad (52)$$

which follows from the identical Larmor radius expansions for the delta-function [Eq. (37)] and the electric field [Eq. (31)]. Integration over canonical angles gives the Fourier coefficient $(\mathcal{E}_\alpha^{(N)})_{\mathbf{m}}^*$, and thus

$$P_{\mathrm{tot}}^{(N)} = \frac{1}{2}\mathrm{Re}\ \sqrt{2\pi}L\frac{e^2}{m_0}\sum_{\mathbf{m}}\int \mathrm{d}r_0 r_0\frac{n_0\omega_c^2}{v_T^5}\int_0^{+\infty}\mathrm{d}\rho\rho \times \mathrm{e}^{-(1/2)(\rho/\rho_L)^2}M^{\alpha\beta}(\tilde{\mathcal{E}}_\alpha^{(N)})_{\mathbf{m}}^*(\tilde{\mathcal{E}}_\beta^{(N)})_{\mathbf{m}}, \quad (53)$$

where we have defined the matrix

$$M^{\alpha\beta} = \int_0^{+\infty}\mathrm{d}\tau\int_{-\infty}^{+\infty}\mathrm{d}u\int_{-\infty}^{+\infty}\mathrm{d}u'\tilde{G}(u,u',\tau) \times \mathrm{e}^{-(1/2)(u'/v_T)^2}[\Omega^\alpha]_u[\Omega^\beta]_{u'}. \quad (54)$$

The matrix $M^{\alpha\beta}$ is symmetric [see Eq. (27)] and, using the polynomial representation of the canonical frequencies [Eqs. (B4) and (B5)], the explicit form is

$$M^{\alpha\beta} = Q_0^\alpha Q_0^\beta W_l^{00} + Q_1^\alpha Q_0^\beta W_l^{10} + Q_0^\alpha Q_1^\beta W_l^{01} + Q_1^\alpha Q_1^\beta W_l^{11}. \quad (55)$$

The sign of the total power absorption [Eq. (53)] is determined by all possible values of the quadratic form

$$\mathcal{F} = \mathrm{Re}(M^{\alpha\beta})[\mathrm{Re}(\tilde{\mathcal{E}}_\alpha^{(N)})_{\mathbf{m}}\mathrm{Re}(\tilde{\mathcal{E}}_\beta^{(N)})_{\mathbf{m}} + \mathrm{Im}(\tilde{\mathcal{E}}_\alpha^{(N)})_{\mathbf{m}}\mathrm{Im}(\tilde{\mathcal{E}}_\beta^{(N)})_{\mathbf{m}}]. \quad (56)$$

To find them, let us introduce new independent variables of the same dimension, $\hat{x}_1=\mathrm{Re}(\tilde{\mathcal{E}}_\phi^{(N)})_{\mathbf{m}}$, $\hat{x}_2=\mathrm{Re}(\tilde{\mathcal{E}}_{\theta_g}^{(N)})_{\mathbf{m}}$, and $\hat{x}_3=r_0\ \mathrm{Re}(\tilde{\mathcal{E}}_{z_g}^{(N)})_{\mathbf{m}}$, and represent the first term in the quadratic form [Eq. (56)] as $\hat{M}^{\alpha\beta}\hat{x}_\alpha\hat{x}_\beta$. To prove that the first term is never negative, it is sufficient to check that all three eigenvalues of the matrix $\hat{M}^{\alpha\beta}$ are nonnegative. The same conclusion is true for the second term in Eq. (56).

It can be shown that the matrix $\hat{M}^{\alpha\beta}$ has only one nonzero eigenvalue,

$$\lambda_3 = \left[ \omega_c^2 + (\sqrt{2}v_T Z_l h^\vartheta + V^\vartheta)^2 + \left( \sqrt{2}v_T Z_l \frac{h^z}{r_0} + \frac{V^z}{r_0} \right)^2 \right] \times \mathrm{Re}\, W_l^{00} \geq 0, \quad (57)$$

$Z_l=(\omega-\omega_l)/(\sqrt{2}k_\parallel v_T)$, and thus the total power absorption by a specific sort of particles is non-negative.

Making use of Eq. (39), the total absorbed power [Eq. (51)] can be represented as

$$P_{\mathrm{tot}}^{(N)} = \int_0^{2\pi} \mathrm{d}\vartheta \int_0^L \mathrm{d}z \int_0^{r_w} \mathrm{d}r r p_{\mathrm{dis}}^{(N)}, \quad (58)$$

where $r_w$ is a plasma radius and

$$p_{\mathrm{dis}}^{(N)} = \frac{1}{2}\mathrm{Re} \sum_{n,n'=0}^{N} \frac{\partial^n \tilde{E}_k^*}{\partial r^n} \sigma_{nn'}^{kl} \frac{\partial^{n'} \tilde{E}_l}{\partial r^{n'}}. \quad (59)$$

The quantity $p_{\mathrm{dis}}^{(N)}$ can be interpreted as a locally dissipated power. The correct definition of a locally dissipated power in inhomogeneous plasma is a subtle point and should be derived within quasilinear theory. For example, we can add to Eq. (59) any function whose integral over the radius is zero. It can be shown that in the collisionless limit expression Eq. (59) coincides with a locally dissipated power derived in quasilinear theory.

The poloidal and toroidal torques acting on the plasma through the Lorenz force of a $(m,n)$ harmonic of the perturbation field are related to the absorbed power $P_{\mathrm{tot}}$ in the whole plasma volume in the following way,[7]

$$T_\vartheta = m\frac{P_{\mathrm{tot}}}{\omega}, \quad T_\varphi = n\frac{P_{\mathrm{tot}}}{\omega}. \quad (60)$$

This relation is valid for any dispersive medium whose response current is linear in the perturbation field and is used in the following for computing the torques.

## III. MODELING RESULTS

First, the results of our previous[7] and present plasma models are compared. For that purpose, for JET-like profiles shown in Fig. 1, we compute so called form-factors[8]

$$T_{m,n}(r) = \frac{B_r^{(plas)}(r)}{B_r^{(vac)}(r)}, \quad (61)$$

where $B_r^{(plas)}$ and $B_r^{(vac)}$ are the amplitudes of the radial magnetic field in plasma and vacuum, respectively. The form-factors $T_{m,n}$ show the screening (or amplifying) effect of plasma response currents on the given $(m,n)$ harmonic of the external perturbation. Although the form-factors have been obtained in cylindrical geometry, they can still be used to estimate the perturbation field in a toroidal plasma if the vacuum perturbation field is known.[8] The conditions prevailing at the various rational surfaces are best characterized by the perpendicular velocities of the plasma species. Therefore, the electron and ion perpendicular velocities (diamagnetic plus $E\times B$) with vertical lines marking the radial position of the rational surfaces are shown in Fig. 2.

Figures 3 and 4 show the form-factors computed by the previous model with the Krook collision operator and by the

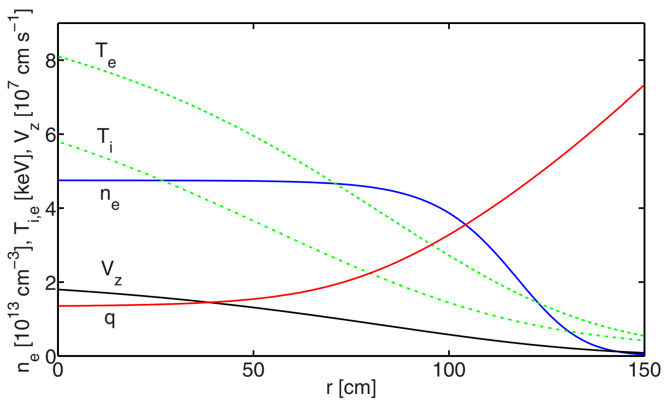


FIG. 1. (Color online) JET-like background profiles of the plasma used in the modeling. Other parameters: $B_0=2$ T (toroidal field on the axis), $R=300$ cm (big radius), poloidal plasma velocity is assumed to be zero. Background magnetic field and plasma current density are computed from the given profiles and equilibrium equations in cylindrical geometry. Background electric field is defined to satisfy the value of perpendicular ion velocity.

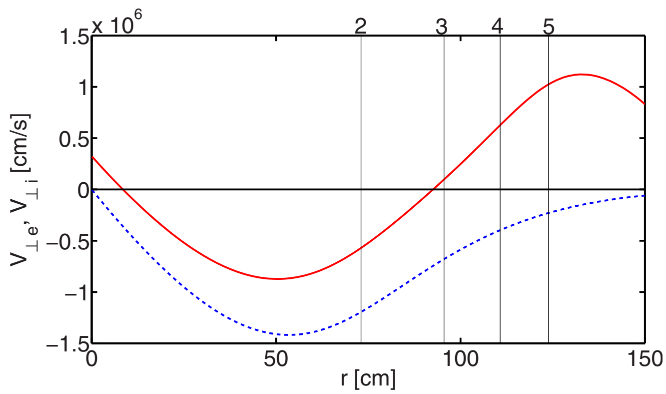


FIG. 2. (Color online) The electron (solid) and ion (dashed) perpendicular velocities (diamagnetic plus $E\times B$), with vertical lines marking the radial position of the rational surfaces $q=2,3,4,5$.

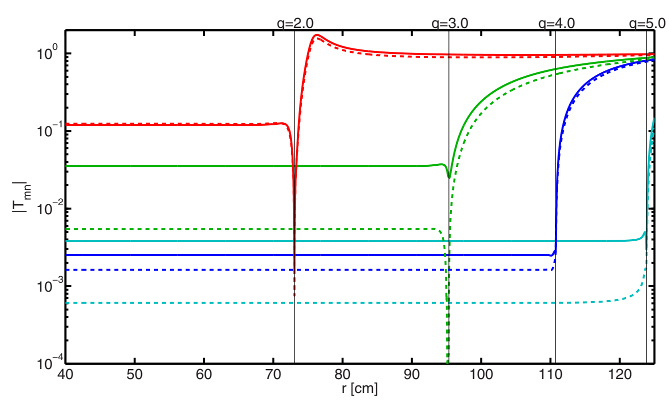


FIG. 3. (Color online) Form-factors computed by previous model in two different frames, moving along $z$-axis with $V=5.0\ 10^9$ cm/s (solid) and $V=-5.0\ 10^9$ cm/s (dashed) for the perturbation harmonics $(-5,1)$, $(-4,1)$, $(-3,1)$, and $(-2,1)$ and frequency $f=10$ Hz. Vertical lines show resonance surfaces with the values of safety factor printed on the upper axis.

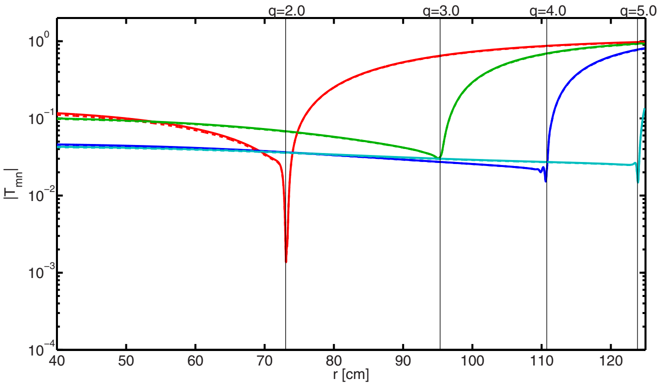


FIG. 4. (Color online) Form-factors as in Fig. 3 computed by the present model.

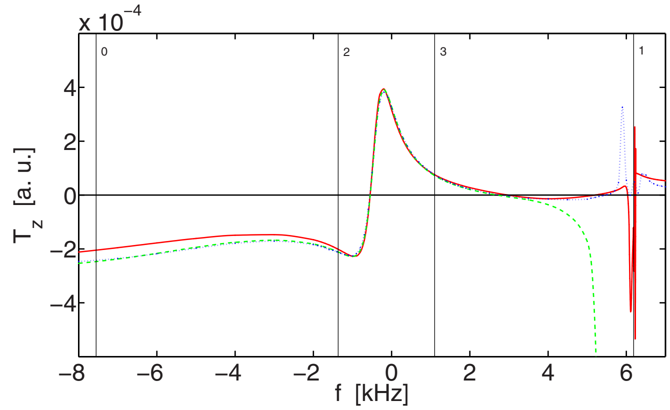


FIG. 6. (Color online) Toroidal torque as a function of the perturbation frequency for the resonant ($m=-9, n=3$) harmonic for $N=1$ (solid), $N=3$ (dashed), and $N=5$ (dotted) orders of the finite Larmor radius expansion. Four vertical lines show the positions of various characteristic electron frequencies: 0 - diamagnetic, 1 - electric rotation $E\times B$, 2 - diamagnetic+$E\times B$, 3 - diamagnetic+$E\times B$+kinetic correction.

present model with the Fokker–Planck collision operator in two different moving frames obtained in first order Larmor radius expansion $N=1$ of the plasma current density [Eq. (39)]. It can be seen that the Galilean invariance of the results is strongly violated for the previous model (especially at the plasma edge where the collision frequency is high), while for the present model, the difference is less than a few percent for all modes of the perturbation. Both models indicate strong shielding (by several orders of magnitude) of the perturbation harmonics at the corresponding resonance surfaces. Nonresonant modes of the perturbation are not affected by currents at resonant surfaces and usually have form-factors around unity. In some cases, they can even be amplified by the plasma response currents.

In the next step, the convergence of the results with respect to higher order Larmor radius expansion is investigated. This is an important issue because the characteristic radial scale of the perturbation fields in the resonance zones is of the order of the ion Larmor radius. As a result, the expansion parameter is of order one and the convergence of the expansion cannot be assured.

Figure 5 shows form-factors for the different orders of Larmor radius expansion $N=1,3,5$. One can see that the form-factors are of the same order of magnitude and that the results for $N=3,5$ are in much better agreement than the results for $N=1,3$. This conclusion is true for all modes except the $(-4,1)$-mode where a fake mode is excited near the resonance region (see comments in Sec. IV). Because the values of the perpendicular electron velocity near the resonant surface $q=3$ are small, the mode (3,1) shielded less than the other modes.

In Figs. 6 and 7, the resonant and nonresonant toroidal torques are shown as functions of the perturbation frequency for different orders of Larmor radius expansion, namely $N=1,3,5$. Figure 6 confirms the well-known fact[7,16,17,21] that the resonant torque basically depends on the value of the characteristic electron frequency at the resonance zone, $\omega_s = k_\parallel V_\parallel + \omega_* + \omega_E - k_\perp T'/(2m_0\omega_c)$, where $\omega_*$ and $\omega_E$ are the frequencies of diamagnetic and electric rotation, $k_\parallel V_\parallel$ is the Doppler shift (negligible), and $k_\perp T'/(2m_0\omega_c)$ is a kinetic correction term. Although a good agreement is obtained for different orders of Larmor radius expansion in a wide frequency range, there exists a region near 6 kHz ($E\times B$ frequency) where strong oscillations of the resonant torque are observed. These oscillations are related to short scale fake

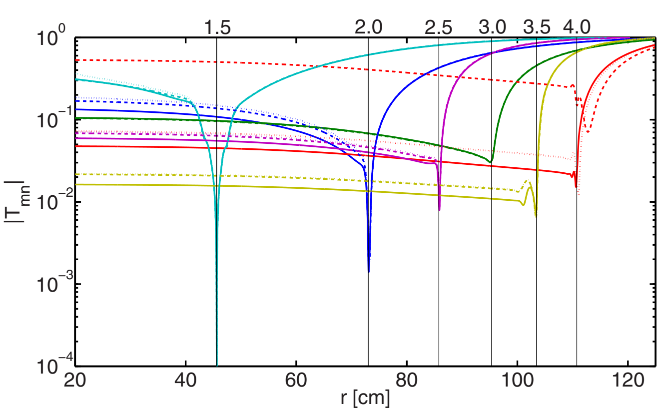


FIG. 5. (Color online) Form-factors computed by present model in different orders of Larmor radius expansion $N=1$ (solid), $N=3$ (dashed), and $N=5$ (dotted) for the perturbation harmonics $(-2,1)$, $(-3,1)$, $(-4,1)$, $(-3,2)$, $(-5,2)$, and $(-7,2)$ and frequency $f=10$ Hz. Vertical lines show resonance surfaces with the values of safety factor printed on the upper axis.

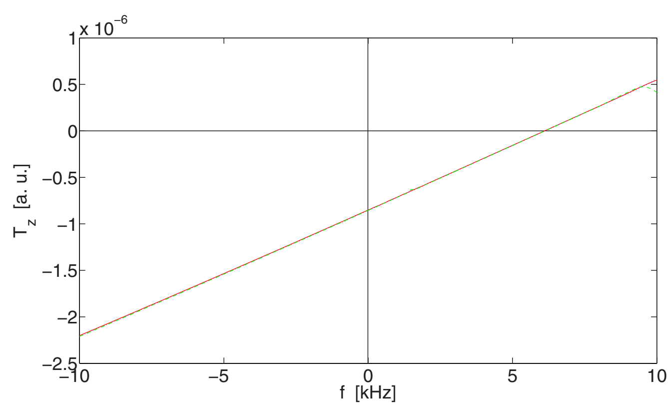


FIG. 7. (Color online) Toroidal torque as a function of the perturbation frequency for the nonresonant ($m=9, n=3$) harmonic for $N=1$ (solid), $N=3$ (dashed), and $N=5$ (dotted) orders of the finite Larmor radius expansion.

modes which are an artifact of the finite Larmor radius expansion. The use of an exact integral operator for the plasma conductivity is needed to validate the results in this region.

## IV. SUMMARY AND DISCUSSION

In this paper, an advanced linear kinetic model of the interaction of helical rotating magnetic perturbations (resonant and nonresonant) with a tokamak plasma is presented. The model is developed in cylindrical geometry and uses a Fokker–Planck type collision operator and a specific Larmor radius expansion scheme of the particle current density. In contrast to the previous paper[7] where Hamiltonian variables had been used for the description of the perturbed quantities, here, the phase space variables are curvilinear coordinates and the covariant components of the *unperturbed* generalized momentum. These variables do not form a canonical conjugate pair with respect to the perturbed Hamiltonian but they are more convenient for the description of the collision operator because they do not contain the perturbation amplitudes.

The present model is implemented in the numerical code named KiLCA (Kinetic Linear Cylindrical Approximation). For a given equilibrium plasma configuration, KiLCA computes the plasma conductivity operator and solves Maxwell equations. Derived quantities like the dissipated power density and the torques acting on the plasma are evaluated in a postprocessing routine within KiLCA.

The present model with Fokker–Planck type collision operator shows a significant improvement of the Galilean covariance in comparison to the previous model[7] with Krook collision term. The use of a finite Larmor radius expansion still makes the Galilean covariance not exact but the level of inaccuracy is now tolerable within a few percent.

The convergence test of the Larmor radius expansion has demonstrated that although the expansion parameter in resonance zones is of order one, the results for different orders of the expansion show good agreement. In some cases, however, the high order expansions produce fake modes in the form of radially short scale slowly damped oscillations. For those artificial modes, the expansion parameter exceeds unity and the resulting perturbation fields are no longer in agreement with the lower order results. Also, due to strong violation of Galilean covariance, those results do depend on the moving frame in which the computations are done and therefore cannot be used. Strictly speaking, the use of the exact integral operator of the plasma conductivity is needed here for an accurate treatment of the RMP problem. In most investigated cases, however, first order finite Larmor radius expansion gives a reasonable approximation and can be used to evaluate form-factors and torques acting on the plasma.

It has been shown in Refs. 7, 16, and 17 that the shielding of RMPs strongly depends on the value of the perturbation frequency evaluated in the frame of reference where the electrons are at rest, $\omega''=\omega-k_\parallel V_\parallel-\omega_*-\omega_E$, where $\omega_*$ and $\omega_E$ are electron frequencies of diamagnetic and electric rotation in the resonance zone. Depending on the plasma configuration and the perturbation amplitude, electromagnetic torques acting on the plasma may significantly change the plasma rotation and thus the value of the parameter $\omega''$ in such a way that the perturbation amplitude inside the plasma strongly increases. Therefore, a self-consistent quasilinear modeling of the interaction of RMPs with the plasma is a necessary further step to find a definite conclusion about the final steady-state shielding of RMPs by the plasma.

## ACKNOWLEDGMENTS

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## APPENDIX A: SOLUTION OF THE KINETIC EQUATION

To simplify the kinetic Eq. (23), we apply a Fourier transform of over $u$ variable

$$\tilde{f}_{\mathbf{m}}(k,t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathrm{d}u\mathrm{e}^{-iku}\tilde{f}_{\mathbf{m}}(u,t),$$

and obtain a first order partial differential equation

$$\left[\frac{\partial}{\partial t}+i\omega_l+Dk_\parallel^2+(\nu k-k_\parallel)\frac{\partial}{\partial k}\right]\tilde{f}_{\mathbf{m}}(k,t)=\tilde{Q}_{\mathbf{m}}(k,t), \qquad (A1)$$

where $\nu=D/v_T^2$ and $\tilde{Q}_{\mathbf{m}}(k,t)$ is a Fourier transform of $\tilde{Q}_{\mathbf{m}}(u,t)$.

The solution of the Eq. (A1) is derived by the characteristics method

$$\tilde{f}_{\mathbf{m}}(k,t)=\int_{t_0}^{t}\mathrm{d}\tau\tilde{Q}_{\mathbf{m}}[\tilde{k}(k,t,\tau-t_0)+k_\parallel/\nu,\tau]\mathrm{e}^{X(k,t,\tau)},$$

where the argument of exponent is defined as

$$X(k,t,\tau)=c(\tau-t)+\tilde{a}(\tau-t)\mathrm{e}^{2\nu(\tau-t)}\tilde{k}^2(k,t,t-t_0)$$
$$+\tilde{b}(\tau-t)\mathrm{e}^{\nu(\tau-t)}\tilde{k}(k,t,t-t_0),$$

$\tilde{k}(k,t,s)=(k-k_\parallel/\nu)\mathrm{e}^{-\nu(t-t_0-s)}$, and $\tilde{a}$, $\tilde{b}$, $c$ functions are defined in Eq. (26).

Taking the inverse Fourier transform of $\tilde{f}_{\mathbf{m}}(k,t)$ we get

$$\tilde{f}_{\mathbf{m}}(u,t)=\int_{t_0}^{t}\mathrm{d}\tau\int_{-\infty}^{+\infty}\mathrm{d}u'\hat{G}(u,u',t-\tau)\tilde{Q}_{\mathbf{m}}(u',\tau),$$

where Green's function is equal to

$$\hat{G}(u,u',\tau)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathrm{d}\hat{k}\exp\left\{i\frac{k_\parallel}{\nu}[u-u']-c(\tau)\right.$$
$$\left.+\hat{k}[i(u-u'\mathrm{e}^{-\nu\tau})-\tilde{b}(\tau)]-\hat{k}^2\tilde{a}(\tau)\right\}.$$

Evaluating the standard Gauss integral, we finally obtain the expression [Eq. (25)].

## APPENDIX B: THE *I* AND *F* FACTORS

In the present model, the unperturbed distribution function is used in the form of an inhomogeneous drifting Maxwellian

$$f_0 = \frac{n_0(r_0)}{[2\pi m_0 T_0(r_0)]^{3/2}} \times \exp\left\{ -\frac{\omega_c(r_0)}{T_0(r_0)} J_\perp - \frac{m_0}{2T_0(r_0)}[u_\parallel - V_\parallel(r_0)]^2 \right\}, \quad \text{(B1)}$$

where the parameters $n_0$, $T_0$, and $V_\parallel$ for each species differ only by first order Larmor radius corrections from the equilibrium density, temperature, and parallel fluid velocity of the respective species. All these parameters and also the equilibrium electrostatic potential $\Phi_0$ are fully defined by the given profiles of plasma density, electron and ion temperatures, poloidal and toroidal plasma velocities, safety factor $q(r)$, and the reference magnetic field value $B_{axis}$.

The Jacobian of the transformation from $(P_\vartheta, P_z)$ to $(r_0, u_\parallel)$ variables [Eq. (21)] is equal to

$$\mathcal{J} = \frac{\partial(P_\vartheta, P_z)}{\partial(r_0, u_\parallel)} = m_0^2 [r_0 \omega_c + (\mathbf{h}')_\perp u_\parallel + (\mathbf{V}'_E)_\perp], \quad \text{(B2)}$$

where $(\mathbf{h}')_\perp = h'_\vartheta h_z - h'_z h_\vartheta$, $(\mathbf{V}'_E)_\perp = V'_{E\vartheta} h_z - V'_{Ez} h_\vartheta$, and $'$ denotes a differentiation over the $r_0$ variable. For the sake of simplicity, below we use only the leading term $m_0^2 r_0 \omega_c$ in the Jacobian, neglecting other terms of order of Larmor radius.

As it follows from Eqs. (44) and (45), all quantities should be finally expressed as functions of the shifted parallel velocity $u = u_\parallel - V_\parallel(r_0)$. Polynomial coefficients $I_\eta^\alpha$ in Eq. (44) are

$$I_\eta^\alpha = m_0^2 r_0 \omega_c Q_\eta^\alpha, \quad \text{(B3)}$$

where $Q_\eta^\phi$ are polynomial coefficients of the canonical frequencies [Eq. (20)], $\Omega^\beta = Q_0^\beta + Q_1^\beta u$, and

$$Q_0^\phi = \omega_c, \quad Q_0^\vartheta = V^\vartheta, \quad Q_0^z = V^z, \quad \text{(B4)}$$

$$Q_1^\phi = 0, \quad Q_1^\vartheta = h^\vartheta, \quad Q_1^z = h^z. \quad \text{(B5)}$$

The derivatives of the background distribution function are

$$\frac{\partial f_0}{\partial J_\perp} = \left(-\frac{f_0}{T_0}\right)\omega_c, \quad \text{(B6)}$$

$$\frac{\partial f_0}{\partial P_\vartheta} = \mathcal{J}^{-1}\left[ m_0 h_z \frac{\partial f_0}{\partial r_0} - m_0(h'_z u_\parallel - \omega_c \hat{h}_\vartheta + V'_{Ez})\frac{\partial f_0}{\partial u_\parallel}\right], \quad \text{(B7)}$$

$$\frac{\partial f_0}{\partial P_z} = \mathcal{J}^{-1}\left[ -m_0 h_\vartheta \frac{\partial f_0}{\partial r_0} + m_0(h'_\vartheta u_\parallel + r_0\omega_c \hat{h}_z + V'_{E\vartheta})\frac{\partial f_0}{\partial u_\parallel}\right], \quad \text{(B8)}$$

where

$$\frac{\partial f_0}{\partial r_0} = \left(-\frac{f_0}{T_0}\right)\left[ -\frac{n'}{n}T_0 + \frac{3}{2}T'_0 + T_0\left(\frac{\omega_c}{T_0}\right)' J_\perp - m_0 V'_\parallel u - \frac{m_0 T'_0}{2T_0}u^2 \right],$$

$$\frac{\partial f_0}{\partial u_\parallel} = \left(-\frac{f_0}{T_0}\right) m_0 u.$$

Derivatives of $f_0$ [Eqs. (B6)–(B8)] can be represented as polynomials in $(J_\perp, u)$ variables

$$\frac{\partial f_0}{\partial J_\beta} = \left(-\frac{f_0}{T_0}\right)[Z_{00}^\beta + Z_{01}^\beta J_\perp + Z_{10}^\beta u + Z_{20}^\beta u^2], \quad \text{(B9)}$$

where

$$Z_{00}^\phi = \omega_c, \quad Z_{00}^\vartheta = \frac{L_1 v_T^2}{r_0 \omega_c} h_z, \quad Z_{00}^z = -\frac{L_1 v_T^2}{r_0\omega_c} h_\vartheta, \quad \text{(B10)}$$

$$Z_{01}^\phi = 0, \quad Z_{01}^\vartheta = \frac{L_2}{r_0 m_0} h_z, \quad Z_{01}^z = -\frac{L_2}{r_0 m_0} h_\vartheta, \quad \text{(B11)}$$

$$Z_{10}^\phi = 0, \quad Z_{10}^\vartheta = h^\vartheta - \frac{V'_z}{r_0\omega_c}, \quad Z_{10}^z = h^z + \frac{V'_\vartheta}{r_0\omega_c}, \quad \text{(B12)}$$

$$Z_{20}^\phi = 0, \quad Z_{20}^\vartheta = -\frac{(h_z v_T)'}{r_0\omega_c v_T}, \quad Z_{20}^z = \frac{(h_\vartheta v_T)'}{r_0 \omega_c v_T}, \quad \text{(B13)}$$

and $L_1 = -n'/n + 3v'_T/v_T$, $L_2 = (v_T^2/\omega_c)(\omega_c/v_T^2)'$.

The term $\mathbf{m}\cdot(\partial f_0/\partial \mathbf{J})$ is

$$\mathbf{m}\cdot\frac{\partial f_0}{\partial \mathbf{J}} = l\frac{\partial f_0}{\partial J_\perp} + \frac{m_0}{\mathcal{J}}\left\{ r_0 k_\perp \frac{\partial f_0}{\partial r_0} + [-(r_0 k_\perp)' u_\parallel + r_0 k_\parallel \omega_c + (r_0 k_\parallel V_E)']\frac{\partial f_0}{\partial u_\parallel}\right\} = \left(-\frac{f_0}{T_0}\right)[P_{00} + P_{01}J_\perp + P_{10}u + P_{20}u^2],$$

where

$$P_{00} = l\omega_c + k_\perp \frac{v_T^2}{\omega_c} L_1, \quad P_{01} = \frac{k_\perp}{m_0} L_2, \quad \text{(B14)}$$

$$P_{10} = k_\parallel - \frac{[r_0(\mathbf{k}\times\mathbf{V})_r]'}{r_0\omega_c}, \quad P_{20} = -\frac{(r_0 v_T k_\perp)'}{r_0 v_T \omega_c}, \quad \text{(B15)}$$

and $\mathbf{k} = k_\perp \mathbf{e}_\perp + k_\parallel \mathbf{h}$, $\mathbf{V} = V_E \mathbf{e}_\perp + V_\parallel \mathbf{h}$.

Making use of $\mathbf{m}\cdot\mathbf{\Omega} - \omega = R_{00} + R_{10}u$, $R_{00} = \omega_l - \omega$, $R_{10} = k_\parallel$, it is straightforward to get polynomial coefficients $F_{\mu\nu}^\beta$ in Eq. (45),

$$F_{00}^\beta = P_{00}Q_{00}^\beta - R_{00}Z_{00}^\beta, \quad F_{01}^\beta = P_{01}Q_{00}^\beta - R_{00}Z_{01}^\beta, \quad \text{(B16)}$$

$$F_{10}^\beta = P_{10}Q_{00}^\beta + P_{00}Q_{10}^\beta - R_{00}Z_{10}^\beta - R_{10}Z_{00}^\beta, \quad \text{(B17)}$$

$$F_{20}^\beta = P_{20}Q_{00}^\beta + P_{10}Q_{10}^\beta - R_{00}Z_{20}^\beta - R_{10}Z_{10}^\beta, \quad \text{(B18)}$$

$$F_{30}^\beta = P_{20}Q_{10}^\beta - R_{10}Z_{20}^\beta, \quad F_{11}^\beta = P_{01}Q_{10}^\beta - R_{10}Z_{01}^\beta. \quad \text{(B19)}$$

### APPENDIX C: EVALUATION OF SPECIAL FUNCTIONS *W* AND *D*

The special function Eq. (47) can be written as

$$W_l^{mn} = \frac{\partial^m}{\partial\alpha^m}\frac{\partial^n}{\partial\beta^n}\int_0^{+\infty} \mathrm{d}\tau \int_{-\infty}^{+\infty} \mathrm{d}u \int_{-\infty}^{+\infty} \mathrm{d}u' \tilde{G}(u,u',\tau) \times \mathrm{e}^{-(1/2)(u'/v_T)^2+\alpha u+\beta u'}\big|_{(\alpha=0,\beta=0)}.$$

As it follows from Eq. (25), the subintegral function is equal to $\mathrm{e}^X/\sqrt{4\pi\tilde{a}}$,

$$X = -\frac{1}{2}\left(\frac{u'}{v_T}\right)^2 + u'[\alpha'\mathrm{e}^{-\nu\tau}+\beta'] - \frac{w^2}{4\tilde{a}} + w\alpha' - i\tilde{b}\alpha' - \tilde{c},$$

where we defined $w=u-u'\mathrm{e}^{-\nu\tau}+i\tilde{b}$, $\alpha'=\alpha+ik_\parallel/\nu$, $\beta'=\beta-ik_\parallel/\nu$. Calculating two standard Gaussian integrals over $u'$ and $w$ variables, we get

$$W_l^{mn} = \sqrt{2\pi}v_T\frac{\partial^m}{\partial\alpha^m}\frac{\partial^n}{\partial\beta^n}\int_0^{+\infty}\mathrm{d}\tau \times \mathrm{e}^{A\tau+B(\alpha,\beta)\mathrm{e}^{-\nu\tau}+C(\alpha,\beta)}\big|_{(\alpha=0,\beta=0)},$$

where

$$A = i(\omega-\omega_l) - \frac{k_\parallel^2 v_T^2}{\nu}, \quad \omega_l = l\omega_c + \omega_E + k_\parallel V_\parallel,$$

$$B = v_T^2\left(\alpha+\frac{ik_\parallel}{\nu}\right)\left(\beta+\frac{ik_\parallel}{\nu}\right),$$

$$C = \frac{1}{2}v_T^2(\alpha^2+\beta^2) - \frac{ik_\parallel}{\nu}v_T^2(\alpha+\beta) + \frac{k_\parallel^2 v_T^2}{\nu^2}.$$

The integration over $\tau$ variable can be reduced to incomplete gamma function and further to confluent hypergeometric function of the first kind,[22] so finally we obtain

$$W_l^{mn} = -\sqrt{2\pi}\frac{v_T}{A}\frac{\partial^m}{\partial\alpha^m}\frac{\partial^n}{\partial\beta^n}\mathrm{e}^{C+B}{}_1F_1\left(1,1-\frac{A}{\nu},-B\right)\big|_{(\alpha=0,\beta=0)}. \quad \text{(C1)}$$

The mentioned symmetry of *W*-function on the upper indices is evident from Eq. (C1). For the computation of numerical values of the ${}_1F_1$ function on a complex plane, we have used continued fractions approximation. In the collisionless limit, $\nu\to 0$,

$$W_l^{00} = \frac{\pi}{k_\parallel}W(Z_l), \quad \text{(C2)}$$

$$W_l^{01} = -\sqrt{2\pi}i\frac{v_T}{k_\parallel}[1+i\sqrt{\pi}Z_l W(Z_l)], \quad \text{(C3)}$$

$$W_l^{02} = \sqrt{2}v_T Z_l W_l^{01}, \quad \text{(C4)}$$

where $W(Z_l)$ is the plasma dispersion function[23] and $Z_l=(\omega-\omega_l)/(\sqrt{2}k_\parallel v_T)$.

For the second special *D*-function, the integral in Eq. (48) can be reduced to the second exponential Weber integral[22] by the differentiation over parameter,

$$\int_0^{+\infty}\mathrm{d}\rho\,\mathrm{e}^{-1/2(\rho/\rho_L)^2}\rho^{2\nu+1}J_l(\kappa\rho)J_l(\tilde{\kappa}\rho) = (-)^\nu\frac{\partial^\nu}{\partial\alpha^\nu}\left[\frac{1}{2\alpha}\exp\left(-\frac{\kappa^2+\tilde{\kappa}^2}{4\alpha}\right)I_l\left(\frac{\kappa\tilde{\kappa}}{2\alpha}\right)\right], \quad \text{(C5)}$$

where after taking the derivative, the parameter $\alpha$ should be set to the value $\alpha=1/(2\rho_L^2)$ and $I_l(x)$ is a modified Bessel function of the first kind.

To avoid typesetting and other errors, the introduced special functions for all relevant indices have been evaluated analytically and exported directly as FORTRAN sources by *Mathematica*® for further use in the wave code.


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